

THE EFFECT OF THE VERTICAL PART OF THE PATH ON THE REAL TIME FEYNMAN RULES IN FINITE TEMPERATURE FIELD THEORY

François Gelis⁽¹⁾

*Laboratoire de Physique Theorique ENSLAPP
URA 1436 du CNRS associée à l'ENS de Lyon*

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Abstract

The effect of the contribution of the vertical part of the real time path is studied completely in the case of two points functions. Indeed, this vertical part generally contributes in the calculation of a given graph. Moreover, this contribution is essential in order to have a consistent equilibrium theory: thanks to this contribution, the Green function are effectively invariant by time translation, as they should be. As a by product, it is shown that the perturbative calculations give a result which does not depend on the initial time t_I and final time t_F of the path. The property of independence with respect to t_I is closely related to the KMS conditions, i.e. to the fact the system is in thermal equilibrium. In the case of two point functions, the contribution of the vertical part can be taken into account by the $n(|k_0|)$ prescription in the usual RTF Feynman rules.

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⁽¹⁾ élève à l'École Normale Supérieure de Lyon. E-mail: fgelis@ens.ens-lyon.fr

I. Introduction

Quantum Field Theory can be formulated on every descending path in complex time plane, provided it begins at an initial time t_I and ends at the time $t_I - i\beta$. Among these paths, of particularly important use are the so-called real-time paths, represented on figure 1. This kind of path has two parts parallel to the real time axis, and two vertical parts. In the following, we will only consider in this family of paths the path corresponding to $\sigma = 0^{(2)}$ (see figure 2), so that there will be only one vertical segment, which will be referred in the following as the "vertical part" and denoted simply C_v instead of C_4 .

It is necessary to find a proper way to get rid of the vertical part if one wants to have simple Feynman rules in Fourier space. The usual way to achieve that [1,2,3,4,5,6,7] has been to simply drop the vertical part, arguing that it does not contribute to the Green functions in the limit $t_I \rightarrow -\infty$, $t_F \rightarrow +\infty$, thanks to the Riemann-Lebesgue lemma which is said to imply the factorization of the generating functional: $Z[j] = Z_{12}[j] Z_{34}[j]$. However, it has been noticed that the vertical part can contribute to the Green functions, for example in vacuum graphs [8,9] and even in every two point function [10]. Alternative paths with only two branches have been introduced [11] in order to justify the 2×2 matrix formalism, but the limiting procedure to obtain Real Time Formalism in Fourier space (simply called RTF in the following) is not completely transparent. In section II, we see by a canonical approach (i.e. without the path integral formalism) how this vertical part does arise, and what is the precise meaning of its suppression. We show that dropping the vertical part is equivalent to perform averages with the free density operator $\exp(-\beta H_0)$ instead of $\exp(-\beta H)$, the dynamics of the fields still being governed by the complete Hamiltonian.

Another problem related to the RTF is the so called $n(|k_0|)$ prescription. Indeed, the free propagators contain the statistical function $n(\cdot)$ always accompanied with a factor $\delta(k_0^2 - \omega_k^2)$, where $\omega_k = \sqrt{\mathbf{k}^2 + m^2}$. So, in the free propagators, it makes no difference to have $n(|k_0|)$ or $n(\omega_k)$. But, because the cancellation of pathologies generates derivatives of the $\delta(\cdot)$ distributions, the choice of the argument of the statistical factors is very important in perturbative calculations where pathologies arise. It has been shown that in order to preserve the KMS relations when the propagators are regularized by a finite parameter $\epsilon > 0$, the $n(|k_0|)$ prescription is *necessary* [3,11].

In [10], it is noticed that the vertical part contributes to every connected two point diagram, and that this contribution can be included among the other terms (those which come from the horizontal parts of the path) by simply substituting in them $n(|k_0|)$ to $n(\omega_k)$. So it seems that the two announced problems are closely related, and that the $n(|k_0|)$ prescription is in fact the consequence of the vertical part contribution. However, the proof given in [10] in order to establish this property for the general two point diagram still uses the limit $t_I \rightarrow -\infty$, $t_F \rightarrow +\infty$ in association with the Riemann-Lebesgue lemma.

In section III of the present paper, we perform simple calculations on the complete real time path, with finite t_I and t_F , and then Fourier transform the result in order to see what is the effect of the vertical part. We reobtain the *result* announced by [10]. Moreover, we also see that the result of the calculation is *independent* of the times t_I and t_F , so that the proofs based on the limits $t_I \rightarrow -\infty$, $t_F \rightarrow +\infty$ seems suspect. In section IV, we prove explicitly that

⁽²⁾ Nevertheless, most of the calculations made in this paper are valid whatever is the path going from t_I to $t_I - i\beta$.

every diagram is independent of t_I and t_F . The independence with respect to t_F is simply related to causality, because t_F must be chosen greater than any other time in the diagram. The independence with respect to t_I is related to the fact that the system is in equilibrium, and is proven by use of the KMS relations.

In section V, we prove that the vertical part contributions can be taken into account by the $n(|k_0|)$ prescription for every two point function, without invoking the $t_I \rightarrow -\infty$ and $t_F \rightarrow +\infty$ limits.

In section VI, we show that it is possible to have a contribution of the vertical part in n point functions ($n > 2$), for certain configurations of the external momenta.

II. Canonical derivation of the generating functional

The path-integral technique, in association with the Feynman-Matthews-Salam formula [3], is the most powerful way to quantize a classical field. Nevertheless, in the case of thermal field theories, a canonical approach can clarify the actual role of the vertical branch of the path. We reproduce here this derivation in the simple case of a real scalar field⁽³⁾ [3,12,13].

2.1 The generating functional for a general average

Let $\phi(x)$ be the field operator in the Heisenberg picture, and $\phi_I(x)$ the field operator in the interaction picture, the interaction being switched on at the time t_I . They are related by:

$$\begin{cases} \phi(x) = U(t_I, t) \phi_I(x) U(t, t_I) \\ U(t_2, t_1) = T_c \exp i \int_{t_1}^{t_2} \mathcal{L}_I(\phi_I(x)) d^4x \end{cases} \quad (1)$$

where the integration is performed along a path going from t_1 to t_2 in the complex-time plane, and where $\mathcal{L}_I(\cdot)$ is the interacting part of the Lagrangian. The operator $U(t_1, t_2)$ verifies the following basic properties:

$$\begin{cases} U(t_1, t_1) = \mathbf{1} \\ U(t_1, t_2) U(t_2, t_3) = U(t_1, t_3) \end{cases} \quad (2)$$

By introducing an arbitrary time t_F , we can write:

$$\phi(x) = U(t_I, t_F) U(t_F, t) \phi_I(x) U(t, t_I) \quad (3)$$

from which we deduce:

$$T_c \phi(x_1) \cdots \phi(x_n) = T_c \left[\phi_I(x_1) \cdots \phi_I(x_n) \exp i \int_{C_1 \oplus C_2} d^4x \mathcal{L}_I(\phi_I(x)) \right] \quad (4)$$

⁽³⁾ Of course, the case of more complicated fields like non abelian gauge fields is prohibitively tedious to treat by this method. But we expect that the interpretation of the vertical part obtained here for scalar fields still holds for other kind of fields.

where C_1 is a path going along the real time axis from t_I to t_F , and C_2 the reverse path from t_F to t_I .

Now, we define the Green functions to be the averages of the path ordered products calculated in (4), an average operator being in general a linear form $\omega(\cdot)$, normalized to $\omega(\mathbf{1}) = 1$. From (4), we can directly write the generating functional of these Green functions:

$$\omega(\text{T}_c \phi(x_1) \cdots \phi(x_n)) = \frac{1}{Z^\omega[0]} \frac{\delta}{i\delta j(x_1)} \cdots \frac{\delta}{i\delta j(x_n)} Z^\omega[j]_{|j=0} \quad (5)$$

$$Z^\omega[j] = \omega \left(\text{T}_c \exp i \int_{C_1 \oplus C_2} d^4x \mathcal{L}_I(\phi_I(x)) + j(x)\phi_I(x) \right) \quad (6)$$

In particular, we will denote:

$$\omega_H(\cdot) \equiv \frac{\text{Tr}(e^{-\beta H} \cdot)}{\text{Tr}(e^{-\beta H})} \quad \omega_{H_0}(\cdot) \equiv \frac{\text{Tr}(e^{-\beta H_0} \cdot)}{\text{Tr}(e^{-\beta H_0})} \quad (7)$$

where H_0 is the free part of the Hamiltonian $H = H_0 + H'$ in the Heisenberg picture.

2.2 The introduction of the vertical part

The fundamental formula [13], useful to perform equilibrium averages (i.e. with $\omega_H(\cdot)$), is:

$$e^{-\beta H} = e^{-\beta H_0} \text{T}_c \exp i \int_{C_v} d^4x \mathcal{L}_I(\phi_I(x)) \quad (8)$$

where C_v is a path going from t_I to $t_I - i\beta$, and which will be taken parallel to the imaginary time axis (figure 2). Then, thanks to the formula (8), we can rewrite the generating functional of the equilibrium Green functions as:

$$Z^H[j] = \frac{\text{Tr}(e^{-\beta H_0})}{\text{Tr}(e^{-\beta H})} \omega_{H_0} \left(\text{T}_c \exp i \int_C d^4x \mathcal{L}_I(\phi_I(x)) + j(x)\phi_I(x) \right) \quad (9)$$

where $C = C_1 \oplus C_2 \oplus C_v$ is the real time path with $\sigma = 0$. If, instead of calculating equilibrium averages, we decide to perform $\omega_{H_0}(\cdot)$ averages, we do not need to use the formula (8) and we can write directly:

$$Z^{H_0}[j] = \omega_{H_0} \left(\text{T}_c \exp i \int_{C_1 \oplus C_2} d^4x \mathcal{L}_I(\phi_I(x)) + j(x)\phi_I(x) \right) \quad (10)$$

So, apart from a j -independent factor which plays no role in the perturbative calculation of a Green function with (5), we see that the only difference between the last two formulae is the integration path. The last one shows that when we neglect the vertical part, we are in reality calculating averages with the free density operator $\exp(-\beta H_0)$.

We can further simplify these generating functionals by the same manipulations as the zero temperature ones [14], and by using the Wick's theorem which remains valid for $\omega_{H_0}(\cdot)$ averages of free fields (see appendix 1). Indeed, we have:

$$\omega_{H_0} \left(T_c \exp i \int_{\gamma} d^4x j(x) \phi_I(x) \right) = \exp -\frac{1}{2} \int_{\gamma} d^4x d^4y j(x) j(y) \omega_{H_0}(T_c \phi_I(x) \phi_I(y)) \quad (11)$$

where γ is an arbitrary path, so (11) applies both to (9) and (10). (11) implies that to evaluate a graph, we need to attach a free propagator $G_0(x, y) = \omega_{H_0}(T_c \phi_I(x) \phi_I(y))$ to each line, and to integrate over the coordinates of each vertex, the integral over time being performed along the path γ .

So, the important conclusion of this part is that the suppression of the vertical part of the time path is equivalent to perform only $\omega_{H_0}(\cdot)$ averages. More precisely, we can say that adding the vertical part allows us to do at the same time the perturbative expansion in the dynamics of the fields (in the same way as in the zero temperature expansion) *and* in their statistics (i.e. in the density operator, which also contain the coupling constant): the vertical part enables us to have a true equilibrium theory *at each order* of the perturbative expansion.

III. A simple example: the insertion of mass terms

3.1 Calculation in time coordinates

3.1.1 Introduction and notations

In order to see what the effect of the vertical part in effective calculations is, we will do the simplest calculation which consists in the convolution of two propagators:

$$F(x_1, x_2) = \int_C d^4x G_0(x_1, x) G_0(x, x_2) \quad (12)$$

This is typically the calculation one would do to insert a mass term on the propagator of a particle. We can further simplify the computations by using a Fourier representation with respect to the spatial dependence:

$$G_0(t_1, t_2; \mathbf{k}) = \int d^3\mathbf{x} e^{-i\mathbf{k}\cdot\mathbf{x}} G_0(t_1, \mathbf{0}, t_2, \mathbf{x}) \quad (13)$$

It is well known that⁽⁴⁾:

$$G_0(t_1, t_2; \omega_k) \equiv G_0(t_1, t_2; \mathbf{k}) = \frac{1}{2\omega_k} \left[e^{-i\omega_k(t_2-t_1)} (\theta_c(t_2 - t_1) + n_B(\omega_k)) + e^{-i\omega_k(t_1-t_2)} (\theta_c(t_1 - t_2) + n_B(\omega_k)) \right] \quad (14)$$

⁽⁴⁾ Whenever a two point function depends on \mathbf{k} only through ω_k , we will recall this by replacing \mathbf{k} by ω_k in the list of its arguments.

So, the Feynman rules will be slightly modified and we have now to compute:

$$F(t_1, t_2; \omega_k) = \int_C dt G_0(t_1, t; \omega_k) G_0(t, t_2; \omega_k) \quad (15)$$

3.1.2 The t_F -independence

Before going to the calculation itself, we make a general remark on the independence of the Green functions with respect to the time t_F . To see this property, we can choose another final time $t'_F > t_F$. Then if G_{t_F} is the Green function calculated on a path which final time is t_F , we will have:

$$G_{t'_F} - G_{t_F} = \int_{\gamma} g(t) dt \quad (16)$$

where γ is a path going from t_F to t'_F and then back to t_F and $g(t)$ a function which does not contain explicitly t_F or t'_F . Moreover, if t_F is greater than all the other times in the problem, $g(t)$ will take the same values on the upper and lower part of γ , and $G_{t'_F}$ will be equal to G_{t_F} , as announced.

3.1.3 Result and commentaries

After some straightforward calculations, we find:

$$\begin{aligned} F(t_1, t_2; \omega_k) = & \frac{1}{4\omega_k^2} \{ n_B^2(\omega_k) [\frac{\beta}{i} (e^{i\omega_k(t_2-t_1)} + e^{-i\omega_k(t_2-t_1)}) \\ & + \frac{1}{2i\omega_k} (e^{i\omega_k(t_1+t_2-2t_I)} (1 - e^{-2\omega_k\beta}) - e^{-i\omega_k(t_1+t_2-2t_I)} (1 - e^{2\omega_k\beta}))] \\ & + n_B(\omega_k) [(t_2 - t_1) (e^{-i\omega_k(t_2-t_1)} - e^{i\omega_k(t_2-t_1)}) \\ & + \left(\frac{1}{i\omega_k} + \frac{\beta}{i} \right) (e^{-i\omega_k(t_2-t_1)} + e^{i\omega_k(t_2-t_1)}) \\ & - \frac{1}{i\omega_k} (e^{i\omega_k(t_1+t_2-2t_I)} e^{-2\omega_k\beta} + e^{-i\omega_k(t_1+t_2-2t_I)})] \\ & + [\theta_c(t_2 - t_1) (t_2 - t_1 + \frac{1}{i\omega_k}) e^{-i\omega_k(t_2-t_1)} + \theta_c(t_1 - t_2) (t_1 - t_2 + \frac{1}{i\omega_k}) e^{i\omega_k(t_2-t_1)} \\ & - \frac{1}{2i\omega_k} (e^{i\omega_k(t_1+t_2-2t_I)} e^{-2\omega_k\beta} + e^{-i\omega_k(t_1+t_2-2t_I)})] \} \end{aligned} \quad (17)$$

Firstly, we notice that there appears some terms which are not invariant by time translation, depending on $t_1 + t_2 - 2t_I$. But, we verify that the sum of all these terms is zero. Therefore, the final result is invariant by time translation and independent of the initial time t_I . We must precise that the contribution of the vertical part⁽⁵⁾ (terms in β/i and in $\exp(\pm 2\beta\omega_k)$) is

⁽⁵⁾ The separation between the contributions of the horizontal part and of the vertical part has no intrinsic meaning: it depends on the position of the external times on the path (if they are all on the vertical part, the entire result comes from the vertical part). So, here and in the following, the contribution of the vertical part is defined to be the one obtained when the external times are on the horizontal branches.

essential in order to have this cancellation. We will generalize this property to a general graph in the next section. Another property of this result is the fact that it remains contributions of the vertical part (terms proportional to β/i), and we have no possibility to make them disappear by the limit procedure $t_I \rightarrow -\infty$, $t_F \rightarrow +\infty$ because the result does not contain t_I or t_F anymore.

We can also verify explicitly the mass-derivative formula [10,15]:

$$i \frac{\partial}{\partial m^2} G_0(t_1, t_2; \omega_k) = \int_C dt G_0(t_1, t; \omega_k) G_0(t, t_2; \omega_k) \quad (18)$$

It is important to notice here that this formula holds only if the integration is performed on the *complete* path C , because the left-hand side of (18) contains those terms in β/i exhibited above.

3.2 The result in Fourier space

Now, the following step is to perform the Fourier transformation of $F(t_1, t_2; \mathbf{k})$ with respect to the difference $t_2 - t_1$ ⁽⁶⁾ in order to see which are the Feynman rules in energy space (they are not obvious because of the contribution of the vertical part in the result). We will denote $D_{ij,\omega}(k)$ the usual real-time matrix propagator (with $\sigma = 0$) with distribution function $n_B(\omega_k)$ and $D_{ij,k_0}(k)$ the same propagators with distribution function $n_B(|k_0|)$ (see appendix 2). These two kinds of *free* propagators are equal because each distribution function is accompanied by a $\delta(k_0^2 - \omega_k^2)$ factor. Then, by a simple but tedious calculation, we find for the Fourier transformation:

$$F_{ij}(k_0, \omega_k) = (D_\omega(k) \tau_3 D_\omega(k))_{ij} + \frac{\beta}{i} \frac{n_B(\omega_k)(1 + n_B(\omega_k))}{2\omega_k} 2\pi \delta(k^2 - m^2) \quad (19)$$

where the last term is the Fourier transformation of the vertical part contribution. Moreover, in the first term, the cancellation of pathologies makes derivatives of delta distributions appear (see appendix 3 and section V), so that this term will not be the same if we replace ω_k by k_0 in it. But, fortunately, if we do such a substitution, we obtain a supplementary term which cancels exactly the second term of (19), so that [10]:

$$F_{ij}(k_0, \omega_k) = (D_{k_0}(k) \tau_3 D_{k_0}(k))_{ij} \quad (20)$$

In this simple example, we have seen that dropping the vertical part contribution is equivalent to use ω -type propagators in the usual real-time Feynman rules [1,2,3], and that we can take this contribution into account by using k_0 -type propagators with the same Feynman rules.

⁽⁶⁾ This Fourier transformation is legitime once we have verified that the result is invariant by time translation. Moreover, the function $F(\cdot)$ being independant of the times t_I and t_F , we are allowed to perform an usual Fourier transformation with times varying from $-\infty$ to $+\infty$. We then define a 2×2 matrix according to the position of the external times on C_1 or C_2 , in the usual way.

3.3 Relation with part II

To end this section, we verify the physical interpretation of section II, i.e. the fact that dropping the vertical part is equivalent to perform averages with the free statistics. For that purpose, we treat perturbatively an additional mass term $-\mu^2\phi^2/2$ (figure 3). If we write the free matrix propagator as:

$$D_{\omega[k_0]}(k) = U_{\omega[k_0]} \begin{pmatrix} i(k^2 - m^2 + i\epsilon)^{-1} & 0 \\ 0 & -i(k^2 - m^2 - i\epsilon)^{-1} \end{pmatrix} U_{\omega[k_0]} \quad (21)$$

where the matrix U contains the distribution functions and verifies $U\tau_3 U = \tau_3$. The only effect of the summation of all the extra mass insertions is to replace m^2 by $m^2 + \mu^2$ in the central matrix. Consider for example the 11 component of the resummed propagator:

$$\mathcal{D}_{11,\omega[k_0]}(k) = \frac{i}{k^2 - m^2 - \mu^2 + i\epsilon} + 2\pi n_B(\omega_k[|k_0|])\delta(k^2 - m^2 - \mu^2) \quad (22)$$

where $\omega_k = \sqrt{k^2 + m^2}$ corresponds to the former mass. We then see that if we use the ω -type free propagators, the result corresponds to the new dynamics (through the change of pole) and to the former statistics (corresponding to the mass m), as announced. If, on the contrary, we use the k_0 -type free propagators, the resultant statistical distribution will be $n_B(\sqrt{k^2 + m^2 + \mu^2})$, i.e. the result takes into account the change of mass also in the statistics.

3.4 Conclusion

On the simple example treated in this part, we have seen that the vertical part does contribute to the final result, and that taking into account this contribution is equivalent to the so-called $n(|k_0|)$ prescription. Moreover, we insist again on the fact that this contribution is independent of the times t_I and t_F . We will extend later this property to a general two point function.

IV. A general proof of the t_I -invariance

4.1 Proof in the case of scalar fields

In the preceding section, we noticed on an example that the result of the time integration along the complete path C was independent of t_I , and was invariant by time translation. Before going on to a general proof of this result, we must emphasize the fact that this is in complete agreement with our hypothesis of thermal equilibrium. Indeed, a system in equilibrium must be invariant by time translation and can't have memory of its initial time: therefore its Green functions must verify also these two properties. We must also insist on their non obvious character: indeed, the free propagators verify them, but the integration path does not⁽⁷⁾.

⁽⁷⁾ As we will see later, we need an extra property of the propagators expliciting the fact that they are equilibrium propagators: i.e. the KMS relations.

In the following, we will denote the value of a n point arbitrary graph by:

$$G_{t_I}(t_1, \dots, t_n) \equiv \int_{C_{t_I}} \prod_{i=1}^v dt_{n+i} F(t_1, \dots, t_n; t_{n+1}, \dots, t_{n+v}) \quad (23)$$

where v is the number of vertices in the graph, and $F(\cdot)$ the product of the free propagators corresponding to the lines of the graph, before we perform the vertex-time integrations⁽⁸⁾. The t_I index is there to recall that the integrations are performed on a path beginning at t_I and ending at $t_I - i\beta$.

Firstly, we can prove the equivalence between the following two properties:

$$\forall(t_1, \dots, t_n), \quad G_{t_I}(t_1, \dots, t_n) = G_{t_I}(t_1 + a, \dots, t_n + a) \quad (24)$$

$$\forall(t_1, \dots, t_n), \quad G_{t_I}(t_1, \dots, t_n) = G_{t_I-a}(t_1, \dots, t_n) \quad (25)$$

because the function $F(\cdot)$ depends only on time-differences (see (14)). Therefore, the invariance by time translation is equivalent to the independence with respect to the initial time t_I . We now prove this last property, first for scalar fields and then for fermions, eventually with a chemical potential.

For that purpose, we need the following two lemmas:

$$\frac{d}{dt_I} \int_{C_{t_I}} f(t) dt = f(t_I - i\beta) - f(t_I) \quad (26)$$

where f is a function defined on the path C_{t_I} , which does not depend explicitly on t_I , and:

$$\begin{cases} G_0(t_I - i\beta, t_2; \omega_k) = G_0(t_I, t_2; \omega_k) \\ G_0(t_1, t_I - i\beta; \omega_k) = G_0(t_1, t_I; \omega_k) \end{cases} \quad (27)$$

The last one is only a consequence of the KMS relations, that is to say of the thermal equilibrium property of the system.

Then, the $F(\cdot)$ function verify also this kind of KMS property:

$$\forall j \in [1, \dots, v], \quad F(\dots, t_{n+j-1}, t_I - i\beta, t_{n+j+1}, \dots) = F(\dots, t_{n+j-1}, t_I, t_{n+j+1}, \dots) \quad (28)$$

It is therefore easy to verify by recursion that the v integrations will lead to a function $G_{t_I}(\cdot)$ which does not depend explicitly on the initial time t_I .

⁽⁸⁾ We have not written the integrations over the three momenta of independent loops, because the spatial dependence of the Green functions has nothing to do with the properties we are looking at here.

4.2 Extension to fermions and chemical potential

The case of fermions with chemical potential is a bit more intricate because the KMS relations are now:

$$\begin{cases} S_0(t_1, t_I; \omega_k) = -e^{-\beta\mu q} S_0(t_1, t_I - i\beta; \omega_k) \\ S_0(t_I, t_2; \omega_k) = -e^{\beta\mu q} S_0(t_I - i\beta, t_2; \omega_k) \end{cases} \quad (29)$$

where μ is the chemical potential and q the conserved charge carried through the propagator from its first point to its second point. Because the number of fermionic lines going to a vertex is even, the minus signs cancel mutually when we try to establish (28) with fermions. In the same way, the $\exp(\pm\beta\mu q)$ factors cancel because of the nullity of the total *conserved* charge arriving at a vertex. So, (28) is still true in the general case of fermions with chemical potential (of course, the KMS relations with respect to the external times depend on the type of the external fields), and that was the only property of the $F(\cdot)$ function we need to prove the invariance of $G_{t_I}(\cdot)$ with respect to t_I .

4.3 Application to ITF

This paragraph lies a bit outside the general line of this article: it shows how the preceding arguments can lead to define "ITF-like" Feynman rules, not only on the usual imaginary time path, but also on every other path going from t_I to $t_I - i\beta$. The property of t_I -invariance can be used to introduce the discrete Matsubara frequencies of the imaginary time formalism. Let's assume that we have performed the calculation of a graph in time coordinates: we are led to a function $G(t_1, \dots, t_n)$ which does not depend on t_I and which verify KMS relations according to the type of fields attached to the external lines. The ITF corresponding function can be defined as [16]:

$$G(\omega_1, \dots, \omega_n) \equiv \int_{G_{t_I}} \prod_{i=1}^n (dt_i e^{-\omega_i t_i}) G(t_1, \dots, t_n) \quad (30)$$

Naturally, we require that this ITF function should be independent of the initial time t_I . It is this requirement that will lead us to the discretization of the Matsubara frequencies. Let's denote:

$$F(t_1, \dots, t_n) \equiv \left(\prod_{i=1}^n e^{-\omega_i t_i} \right) G(t_1, \dots, t_n) \quad (31)$$

According to the proof given in the preceding paragraphs, the ITF function will be independent of t_I if the function $F(\cdot)$ verify scalar-like KMS relations:

$$\forall j \in [1, \dots, n], \quad F(\dots, t_{j-1}, t_I, t_{j+1}, \dots) = F(\dots, t_{j-1}, t_I - i\beta, t_{j+1}, \dots) \quad (32)$$

This lead us to the usual frequencies:

(i) if the line $n^\circ i$ carries a real scalar:

$$\omega_i = \frac{2\pi n_i}{\beta} \quad \text{with} \quad n_i \in \mathbf{Z} \quad (33)$$

(ii) if the line $n^\circ i$ carries a fermion:

$$\omega_i = \frac{(2n_i + 1)\pi}{\beta} \quad \text{with} \quad n_i \in \mathbf{Z} \quad (34)$$

(iii) if the line $n^\circ i$ carries a fermion with conserved charge q_i :

$$\omega_i = \frac{(2n_i + 1)\pi}{\beta} - i\mu q_i \quad \text{with} \quad n_i \in \mathbf{Z} \quad (35)$$

It is then easy to prove the following property [16]:

$$\mathbf{G}(\omega_1, \dots, \omega_n) = -i\beta\delta_{\Sigma,0}\tilde{\mathbf{G}}(\omega_2, \dots, \omega_n) \quad (36)$$

with

$$\tilde{\mathbf{G}}(\omega_2, \dots, \omega_n) = \int_{\tilde{C}_{t_I}} \prod_{i=2}^n (dt_i e^{-\omega_i t_i}) G(0, t_2, \dots, t_n) \quad (37)$$

where $\Sigma = \omega_1 + \dots + \omega_n$ is the total energy entering the graph. We can note that the invariance of the result of time integrations with respect to t_I is essential in this proof. This is quite normal because the t_I -invariance is related to the invariance by time translation, which is itself related to the conservation of total energy we want to prove here.

The inverse transformation is:

$$G(t_1, \dots, t_n) = \left(\frac{i}{\beta}\right)^n \sum_{n_1 \dots n_n \in \mathbf{Z}} \left(\prod_{i=1}^n e^{\omega_i t_i}\right) \mathbf{G}(\omega_1, \dots, \omega_n) \quad (38)$$

and we have the following properties:

$$H(t_1, t_2) = \int_{\tilde{C}_{t_I}} dt F(t_1, t) G(t, t_2) \Leftrightarrow \mathbf{H}(\omega_1, \omega_2) = \frac{i}{\beta} \sum_{n \in \mathbf{Z}} \mathbf{F}(\omega_1, \omega) \mathbf{G}(-\omega, \omega_2) \quad (39)$$

$$H(t_1, t_2) = F(t_1, t_2) G(t_1, t_2) \Leftrightarrow \mathbf{H}(\omega_1, \omega_2) = \left(\frac{i}{\beta}\right)^2 \sum_{n'_1, n'_2 \in \mathbf{Z}} \mathbf{F}(\omega'_1, \omega'_2) \mathbf{G}(\omega_1 - \omega'_1, \omega_2 - \omega'_2) \quad (40)$$

The transformed free scalar propagator is:

$$\mathbf{D}_0(\omega_1, \omega_2) = \frac{\beta}{i} \delta_{\omega_1 + \omega_2, 0} \frac{1}{(i\omega_1)^2 - \omega_k^2} \quad (41)$$

All these relations enable us to use the usual "ITF" Feynman rules. The important point to note here is the fact that these Feynman rules are not at all dependant of the choice of the path going from t_I to $t_I - i\beta$, so that the name "Imaginary Time Formalism" gives a very restrictive view of this formalism.

4.4 Conclusion

The conclusion of this part is that every graph evaluated on the complete path C is invariant by time translation and does not depend explicitly on the initial time t_I . In the proof, this property has been shown to be closely related with thermal equilibrium through the KMS relations. As a corollary of this result, we can look with suspicion the arguments based on the limits $t_I \rightarrow -\infty$, $t_F \rightarrow +\infty$ because nothing special occurs when we look at these limits.

V. The vertical part contribution for two point functions

5.1 Spectral representation of two points functions

In order to extend the result of part III to general two point functions, we use their spectral representation. Let $\mathcal{F}(t_1, t_2; \mathbf{k})$ be a two point function; we can write it as [3, 10] :

$$\mathcal{F}(t_1, t_2; \mathbf{k}) = \int_0^{+\infty} dE f(E, \mathbf{k}) G_0(t_1, t_2; E) \quad (42)$$

where $f(E, \mathbf{k})$ is its spectral density and $G_0(\cdot)$ the free propagator. In particular, this is true for a self-energy function:

$$\Sigma(t_1, t_2; \mathbf{k}) = \int_0^{+\infty} dE \sigma(E, \mathbf{k}) G_0(t_1, t_2; E) \quad (43)$$

Then, if Σ_{ij} is the matrix self-energy, we have:

$$\left\{ \begin{array}{ll} \Sigma_{11}(k_0, \mathbf{k}) = \int_0^{+\infty} dE \sigma(E, \mathbf{k}) D_{11}(k_0, E) & \Sigma_{22}(k_0, \mathbf{k}) = \int_0^{+\infty} dE \sigma(E, \mathbf{k}) D_{22}(k_0, E) \\ \Sigma_{12}(k_0, \mathbf{k}) = - \int_0^{+\infty} dE \sigma(E, \mathbf{k}) D_{12}(k_0, E) & \Sigma_{21}(k_0, \mathbf{k}) = - \int_0^{+\infty} dE \sigma(E, \mathbf{k}) D_{21}(k_0, E) \end{array} \right. \quad (44)$$

where the minus signs in the 12 and 21 components are due to the opposite sign of the type 2 vertices.

5.2 The insertion of a self-energy term

5.2.1 Calculation in time coordinates

We now consider the insertion of a self-energy type function between two free propagators⁽⁹⁾ (figure 4):

$$F(t_1, t_2, \mathbf{k}) \equiv \int_C dt_3 dt_4 G_0(t_1, t_3; \omega_k) (-i\Sigma(t_3, t_4; \mathbf{k})) G_0(t_4, t_2; \omega_k) \quad (45)$$

Thanks to the spectral representation of $\Sigma(\cdot)$, this calculation is simply the convolution of three free propagators. In the following, we need only the contribution of the vertical part, which is:

$$F_v(t_1, t_2; \mathbf{k}) = \frac{\beta n_B(\omega_k)(1 + n_B(\omega_k))}{i(2\omega_k)^2} \sum_{\epsilon, \eta = \pm 1} e^{-i\eta\omega_k(t_2 - t_1)} \int_0^{+\infty} dE \frac{\epsilon\sigma(E, \mathbf{k})}{2E} \frac{\mathcal{P}}{\eta\omega_k - \epsilon E} \quad (46)$$

Its Fourier transformation is independant of the ij component considered because $F_v(\cdot)$ does not contain any $\theta_c(\cdot)$ function. We have:

$$F_{v,ij}(k_0, \mathbf{k}) = \frac{\beta n_B(\omega_k)(1 + n_B(\omega_k))}{i2\omega_k} 2\pi\delta(k_0^2 - \omega_k^2) \int_0^{+\infty} dE \sigma(E, \mathbf{k}) \frac{\mathcal{P}}{k_0^2 - E^2} \quad (47)$$

5.2.2 Calculation in RTF

Now, we do the same calculation with the usual Feynman rules of RTF in order to see the difference between the $n_B(\omega_k)$ and the $n_B(|k_0|)$ prescriptions. We have to evaluate:

$$F_{\omega[k_0],ij}(k_0, \mathbf{k}) \equiv \sum_{a,b=1,2} D_{\omega[k_0],ia}(k_0, \omega_k) (-i\Sigma_{ab}(k_0, \mathbf{k})) D_{\omega[k_0],bj}(k_0, \omega_k) \quad (48)$$

If we explicit a bit more this quantity, we see that the ill defined products of distributions cancels mutually thanks to the following identities:

$$\begin{cases} \Sigma_{11} + \Sigma_{22} + \Sigma_{12} + \Sigma_{21} = 0 \\ (\theta(-k_0) + n_B(|k_0|))\Sigma_{21}(k_0, \mathbf{k}) = (\theta(k_0) + n_B(|k_0|))\Sigma_{12}(k_0, \mathbf{k}) \end{cases} \quad (49)$$

(The first one is always true, and the second one, being a form of the KMS relation, is only true for equilibrium).

Moreover, this cancellation makes derivatives of $\delta(\cdot)$ functions appear. For exemple, the 11 component is:

$$\begin{aligned} F_{\omega[k_0],11}(k_0, \mathbf{k}) &= \frac{i\mathcal{P}}{(k^2 - m^2)^2} \Sigma_{11}(k_0, \mathbf{k}) \\ &\quad - \pi [(1 + 2n_B)\Sigma_{11} + (\theta(-k_0) + n_B)\Sigma_{21} + (\theta(k_0) + n_B)\Sigma_{12}] \frac{\partial}{\partial k_0^2} \delta(k_0^2 - \omega_k^2) \end{aligned} \quad (50)$$

⁽⁹⁾ This operation is the basis of the construction of the complete propagator.

Therefore, the difference $F_{k_0,ij}(k_0, \mathbf{k}) - F_{\omega,ij}(k_0, \mathbf{k})$ will not be zero (see appendix 3). By an explicit calculation, we verify that this difference is independant of the component ij one is considering:

$$F_{k_0,ij}(k_0, \mathbf{k}) - F_{\omega,ij}(k_0, \mathbf{k}) = \frac{\beta}{i} \frac{n_B(\omega_k)(1 + n_B(\omega_k))}{2\omega_k} 2\pi\delta(k_0^2 - \omega_k^2) \left[-i \left(\Sigma_{11} + \frac{\Sigma_{12} + \Sigma_{21}}{2} \right) \right] \quad (51)$$

5.3 Epilogue

Then, thanks to the spectral representation of the self-energy matrix and by using the explicit form of the free matrix propagator (see appendix 2), we can verify the identity:

$$\Sigma_{11} + \frac{\Sigma_{12} + \Sigma_{21}}{2} = \frac{\Sigma_{11} - \Sigma_{22}}{2} = \int_0^{+\infty} dE \sigma(E, \mathbf{k}) \frac{i\mathcal{P}}{k_0^2 - E^2} \quad (52)$$

So that we have:

$$F_{v,ij}(k_0, \mathbf{k}) = F_{k_0,ij}(k_0, \mathbf{k}) - F_{\omega,ij}(k_0, \mathbf{k}) \quad (53)$$

So, we have proven that the contribution of the vertical part corresponds exactly to the difference between the results given by the $n_B(|k_0|)$ and the $n_B(\omega_k)$ prescriptions.

Therefore, to take this vertical part into account, we have to apply the $n_B(|k_0|)$ prescription in the usual Real Time Feynman rules⁽¹⁰⁾, at least for two point functions.

VI. General remarks on n point graphs

6.1 Preliminaries

Before going to the case of higher n point functions, let's consider the following integral⁽¹¹⁾:

$$\mathcal{F}(\Sigma) = \int_C dt F(t) e^{-i\Sigma t} \quad (54)$$

where $\Sigma \in \mathbf{R}$ and where $F(\cdot)$ is the characteristic function of a subset \mathcal{A} of C ($F(t) = 0$ if $t \notin \mathcal{A}$, $F(t) = 1$ if $t \in \mathcal{A}$), with $\mathcal{A} \cap C_v = \emptyset$ or C_v . We will explain later the reason of this condition. Our purpose is here to determine the contribution of the vertical part in $\mathcal{F}(\cdot)$. If

⁽¹⁰⁾ Here, we admit that the contribution of the two horizontal parts is obtained by the usual RTF Feynman rules with $n_B(\omega_k)$ prescription. This point is rather obvious because the propagators in time coordinates already contains $n_B(\omega_k)$ distribution functions.

⁽¹¹⁾ This is the kind of calculation we have to perform in time coordinates, if we except the three momentum integrations in loops.

$\mathcal{A} \cap C_v = \emptyset$, this contribution is trivially zero, so that we only consider the other case. We have now to distinguish between $\Sigma \neq 0$ and $\Sigma = 0$:

(i) if $\Sigma \neq 0$, we have:

$$\mathcal{F}(\Sigma) = \frac{i}{\Sigma} e^{-i\Sigma t_I} (e^{-\Sigma\beta} - 1) + \int_{C_1 \oplus C_2} dt F(t) e^{-i\Sigma t} \quad (55)$$

But we know that the result of the calculation of the graph, which is a sum of such terms, is independent of t_I . So, there must exist other terms which cancel the contribution of the vertical part in (55), because it depends on t_I . The conclusion is that when $\Sigma \neq 0$, (54) will not give a vertical part contribution to the graph.

(ii) if $\Sigma = 0$, we have:

$$\mathcal{F}(\Sigma) = -i\beta + \int_{C_1 \oplus C_2} dt F(t) e^{-i\Sigma t} \quad (56)$$

Now, the preceding argument cannot be applied because the contribution of the vertical part does not depend on t_I . So, in general, there is a contribution of the vertical part in (54) to the graph when $\Sigma = 0$.

6.2 Application to n point functions

The calculation of the vertical part contribution to the time dependence of a n point function can be reduced to a sum of integrals like (54), where $F(\cdot)$ is a product of $\theta_c(\cdot)$ functions and Σ a linear combination of the energies $\omega_i = \sqrt{m^2 + \mathbf{k}_i^2}$ carried by the various lines of the graph. The hypothesis made above on $\mathcal{A} \cap C_v$ is related to the fact that we have defined the contribution of the vertical part to be the one obtained when the integration time is on C_v and the other times on $C_1 \oplus C_2$ (see note 5).

So, we have shown that the result of a vertex integration yields a vertical part contribution if and only if its Σ is zero. Moreover, this contribution is always finite ($-i\beta$ multiplied by a regular function of the other times). Then, if we take into account the fact that we have to perform integrations over the internal independent three momenta, this contribution will vanish at the end when Σ contains an internal momentum because $\Sigma = 0$ defines a null measure subset of the set in which this momentum can vary.

The only remaining possibility to have contributions of the vertical part is then in terms for which Σ contains only the *external* three momenta, or if the diagram contains a self-energy subdiagram (this situation has been studied in the preceding section).

6.3 Example

In order to illustrate this, we consider the example of figure 5. The external propagators, carrying energy ω_i , are denoted for example:

$$G_0(t_1, u_1; \omega_1) = \frac{1}{2\omega_1} \sum_{\epsilon_1 = \pm 1} [\theta_c(\epsilon_1(u_1 - t_1)) + n_B(\omega_1)] e^{-i\epsilon_1 \omega_1 (u_1 - t_1)} \quad (57)$$

and the internal ones are denoted for example:

$$G_0(u_1, u_2; \Omega_1) = \frac{1}{2\Omega_1} \sum_{\eta_1=\pm 1} [\theta_c(\eta_1(u_2 - u_1)) + n_B(\Omega_1)] e^{-i\eta_1\Omega_1(u_2 - u_1)} \quad (58)$$

So, we have to perform an integral like:

$$\int_C \prod_{i=1}^5 (du_i e^{-i\Sigma_i u_i}) F(t_1, t_2, t_3; u_1, u_2, u_3, u_4, u_5) \quad (59)$$

with:

$$\begin{cases} \Sigma_1 = \epsilon_1\omega_1 - \eta_1\Omega_1 + \eta_6\Omega_6, & \Sigma_2 = \epsilon_2\omega_2 - \eta_2\Omega_2 + \eta_1\Omega_1, & \Sigma_3 = \epsilon_3\omega_3 - \eta_3\Omega_3 + \eta_2\Omega_2 \\ \Sigma_4 = \eta_3\Omega_3 - \eta_4\Omega_4 - \eta_5\Omega_5, & \Sigma_5 = \eta_4\Omega_4 + \eta_5\Omega_5 - \eta_6\Omega_6 \end{cases} \quad (60)$$

At the first integration (over u_1), the Σ introduced above is Σ_1 . At the second integration, it can be Σ_2 or $\Sigma_1 + \Sigma_2$ because the first integration can generate a term in $\exp(-i\Sigma_1 u_2)$. At the third integration, it can be Σ_3 , $\Sigma_1 + \Sigma_3$, $\Sigma_2 + \Sigma_3$ or $\Sigma_1 + \Sigma_2 + \Sigma_3$, and so on. Among all the possible Σ encountered in the five integrations, only one depends only on the external energies ω_i :

$$\Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4 + \Sigma_5 = \sum_{i=1}^3 \epsilon_i \omega_i \quad (61)$$

which is encountered in the last integration. Moreover, because of the conservation of three momentum, we have $\Omega_3 = \Omega_6$ so that:

$$\Sigma_4 + \Sigma_5 = (\eta_3 - \eta_6)\Omega_3 \quad (62)$$

and can be zero independantly of Ω_3 when $\eta_3 = \eta_6$: this is precisely what occurs when a subdiagram is of self-energy type.

6.4 Conclusion

The conclusion of this section is that it is possible to have a contribution of the vertical part of the path in $n > 2$ point functions for certain configurations of the external momenta (namely, when a combination $\sum_{i=1}^n \epsilon_i \omega_i$ of the external energies is zero) or when the graph contains a self-energy subdiagram.

VII. Concluding remarks

The main result of this paper is to explicit the role of the vertical part of the real time path, giving a rather physical interpretation of its role. We proved that the vertical part is necessary in order to have the cancellation of the terms which are not invariant by time translation, and that the remaining terms do not depend on the initial time choosen for the path. We must emphasize that such a property was to be expected on physical considerations for an equilibrium system. As a consequence, the usual proofs of the Fourier Feynman rules based on the limits $t_I \rightarrow -\infty$, $t_F \rightarrow +\infty$ are thought to be false.

The other result is that the vertical part of the path generally contributes to the Green functions, as already seen in [10]. Moreover, we have shown in the case of two points functions that this contribution is simply taken into account by the $n_B(|k_0|)$ prescription associated to the usual Real Time Feynman rules. This proof is consistent with the fact that the Green functions does not depend on the contour parameters, because it works even if we keep t_I finite.

It remains to study more precisely the case of the functions with $n > 2$ points, and to see if the contribution of the vertical part which can occur can be dealt simply by the $n_B(|k_0|)$ prescription. Another problem is to make more precise the relation between pathologies and the fact that the theory is or is not an equilibrium one, as outlined in [17,18].

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Note added in proof:

After this work had been completed, we became aware of a preprint by T.S. Evans and A.C. Pearson [19], which contains a rather different approach to the problem of the vertical parts of the path in RTF.

In [19], an ϵ -regularized propagator is used:

$$G_0^\epsilon(t_1, t_2; \omega_k) \equiv G_0(t_1, t_2; \omega_k) e^{-\epsilon|t_2-t_1|} \quad (63)$$

which vanishes when the time difference goes to infinity, in association with the limit $t_I \rightarrow -\infty$. In order to clarify the contradiction between their result and ours concerning the contribution of the vertical part, we repeated the simple calculation presented in section III of the present paper with their regularized propagator, and we were led to the conclusion that the limits $\epsilon \rightarrow 0^+$ and $t_I \rightarrow -\infty$ do not commute because of terms like $\exp(\epsilon t_I)$. More precisely:

(i) If we do $\epsilon \rightarrow 0^+$ first (this is equivalent to the use of non-regularized propagators), the result is invariant by time translation and does not depend anymore on t_I , so that the limit $t_I \rightarrow -\infty$ is useless. Moreover, there remains a contribution of the vertical part of the path in the result.

(ii) If we do $t_I \rightarrow -\infty$ first, like in [19], the result is also invariant by time translation (but this time for different reasons, because the free regularized propagator does not verify KMS) but the contribution of the vertical part is lost.

Because of the physical interpretation of thermal equilibrium, we believe that in a consistent answer to this problem the limit $t_I \rightarrow -\infty$ has no role to play, so that in time coordinates, the use of non-regularized propagators seems preferable.

Appendix 1 : The generating functional

Our purpose is to transform expressions like:

$$\begin{aligned} Z^\omega[j] &= \omega \left(T_c \exp i \int_\gamma d^4x \mathcal{L}_I(\phi_I(x)) + j(x)\phi_I(x) \right) \\ &= \exp i \int_\gamma d^4x \mathcal{L}_I \left(\frac{\partial}{i\partial j(x)} \right) \omega \left(T_c \exp i \int_\gamma d^4x j(x)\phi_I(x) \right) \end{aligned} \quad (A1)$$

To achieve that, we will use twice the Campbell-Haussdorf formula in order to transform the path ordering in normal ordering [14]. By doing this, we make a c-number appear, and the main point is that a c-number is equal to its average $c = \omega_1(c)$, where $\omega_1(\cdot)$ is another arbitrary average operator. The general result is then (in the case of scalar fields):

$$Z^\omega[j] = \exp i \int_\gamma d^4x \mathcal{L}_I \left(\frac{\partial}{i\partial j(x)} \right) \times Z_0^\omega[j] \quad (A2)$$

with

$$Z_0^\omega[j] = C[j] \times \exp -\frac{1}{2} \int_\gamma d^4x d^4y j(x)j(y) \omega_1(T_c \phi_I(x)\phi_I(y)) \quad (A3)$$

$$\begin{aligned} C[j] &= \omega \left(: \exp i \int_\gamma d^4x j(x)\phi_I(x) : \right) \\ &\quad \times \exp \frac{1}{2} \int_\gamma d^4x d^4y j(x)j(y) \omega_1 \left(\phi_I^+(x)\phi_I^-(y) + \phi_I^+(y)\phi_I^-(x) \right. \\ &\quad \left. + \phi_I^+(x)\phi_I^+(y) + \phi_I^-(x)\phi_I^-(y) \right) \end{aligned} \quad (A4)$$

where ϕ_I^+ and ϕ_I^- are respectively the creation and the annihilation part of the free field⁽¹⁾. The first factor in A2 will be expanded in powers of the coupling constant in order to perform the perturbative expansion. The factor $C[j]$ contains the information relative to the initial correlations of the system: we have $C[j] = 1$ when the Wick's theorem holds. The equilibrium field theory is a bit particular because we can replace $\omega_H(\cdot)$ by $\omega_{H_0}(\cdot)$ in $C[j]$ provided we

⁽¹⁾ This expression for $Z_0^\omega[j]$ is consistent with the Dyson-Schwinger equation for real scalar fields [3]:

$$(\square_x + m^2) \frac{\delta Z_0^\omega[j]}{i\delta j(x)} = j(x) Z_0^\omega[j]$$

add the vertical part C_v to the path $C_1 \oplus C_2$. Then, by choosing $\omega_1 = \omega_{H_0}^{(2)}$, we can make the correlation terms disappear⁽³⁾, so that the Feynman rules will be the usual ones, the only difference being the integration path.

We can add that in the general case, there is no simple choice of the path and of the auxiliary average allowing $C[j] = 1$. The consequence of this impossibility is that one will have very intricate Feynman rules with not only propagators, but also correlation functions [20, 21, 22].

Appendix 2 : The free RTF propagators

Here we recall the free matricial propagators we use in RTF with $\sigma = 0$:

$$\begin{cases} D_{\omega[k_0],11}(k_0, \omega_k) = \frac{i\mathcal{P}}{k_0^2 - \omega_k^2} + \pi(1 + 2n_B(\omega_k[|k_0|]))\delta(k_0^2 - \omega_k^2) \\ D_{\omega[k_0],22}(k_0, \omega_k) = \frac{-i\mathcal{P}}{k_0^2 - \omega_k^2} + \pi(1 + 2n_B(\omega_k[|k_0|]))\delta(k_0^2 - \omega_k^2) \\ D_{\omega[k_0],12}(k_0, \omega_k) = 2\pi(\theta(-k_0) + n_B(\omega_k[|k_0|]))\delta(k_0^2 - \omega_k^2) \\ D_{\omega[k_0],21}(k_0, \omega_k) = 2\pi(\theta(k_0) + n_B(\omega_k[|k_0|]))\delta(k_0^2 - \omega_k^2) \end{cases} \quad (A5)$$

where the argument of the distribution functions is ω_k or $|k_0|$, according to the prescription we choose.

Appendix 3 : Distributions and pathologies

The distributions we have to deal with in this paper are $\delta(x)$ and \mathcal{P}/x . The squares $\delta^2(x)$ and $(\mathcal{P}/x)^2$ are ill defined, even in terms of distributions, but it is possible to give a sense to a linear combination of them. To do so, the proper method is to regularize the distributions we are interested in. A simple regularization is given by [3]:

$$\lim_{\epsilon \rightarrow 0^+} \left[\frac{1}{x + i\epsilon} \right]^{n+1} = \frac{\mathcal{P}}{x^{n+1}} + i \frac{\pi(-1)^{n+1}}{n!} \frac{d^n}{dx^n} \delta(x) \quad (A6)$$

where the derivative is defined in the sense of distributions:

$$\forall f, \quad D'(f) = -D(f') \quad (A7)$$

By using (A5) for $n = 0$ and $n = 1$, we obtain the following relations:

$$\frac{\mathcal{P}}{x^2} = \left(\frac{\mathcal{P}}{x} \right)^2 - \pi^2 \delta^2(x) \quad (A8)$$

⁽²⁾ The fact that we can make $C[j] = 1$ with $\omega_1 = \omega_{H_0}$ is very important to have a consistent free theory. Indeed, $\omega_{H_0}(\mathcal{T}_c \phi_I(x) \phi_I(y))$ is a free quantity both for the dynamics of the fields and for the density operator with which the average is performed.

⁽³⁾ The denomination "correlation terms" is a bit ambiguous in this context because they depend on the the actual choice of the integration path and of the auxiliary average $\omega_1(\cdot)$.

$$2\delta(x)\frac{\mathcal{P}}{x} = -\delta'(x) \quad (A9)$$

These relations, combined to the definition of the distributional derivative, imply that the prescription for the argument in the statistical factors is important each time we have to multiply free propagators carrying the same momentum. Indeed, if we look at a product like $a(x)\delta(x)(\mathcal{P}/x)$, the result will be a different distribution if we replace x by 0 in $a(\cdot)$ ⁽⁴⁾. In order to see that⁽⁵⁾, consider the distribution $(a(x) - a(0))\delta'(x)$ and apply it to a test function $f(x)$; the result is not zero:

$$\begin{aligned} \int_{-\infty}^{+\infty} dx (a(x) - a(0))\delta'(x)f(x) &= - \int_{-\infty}^{+\infty} dx \delta(x)[(a(x) - a(0))f(x)]' \\ &= -a'(0)f(0) \neq 0 \end{aligned} \quad (A10)$$

⁽⁴⁾ Of course, we have $a(x)\delta(x) = a(0)\delta(x)$. In fact, the problem comes from the fact that we cannot define unambiguously an associative multiplication of distributions.

⁽⁵⁾ Another way to see the problem is to perform the inverse Fourier transformation of a function where pathologies does arise. Then, one will encounter double poles, the residue of which contains derivatives: the various prescriptions for the arguments of the distribution functions will lead us to different results.

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FIGURE CAPTIONS

- Figure 1** The Real Time Path
- Figure 2** The Real Time Path with $\sigma = 0$
- Figure 3** Repeated insertions of mass terms
- Figure 4** Insertion of a self-energy function
- Figure 5** An example of 3 point diagram

FIGURES

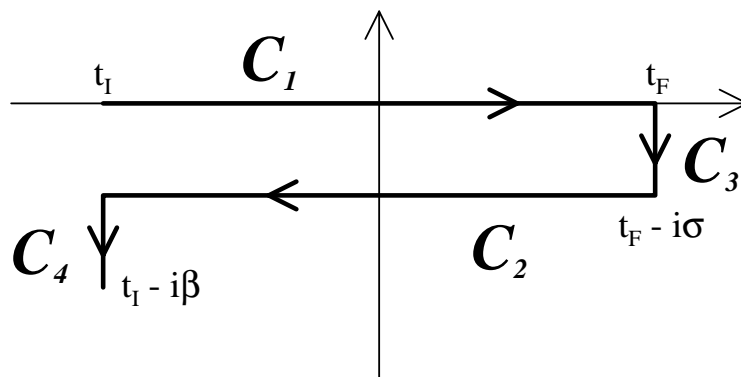


Figure 1

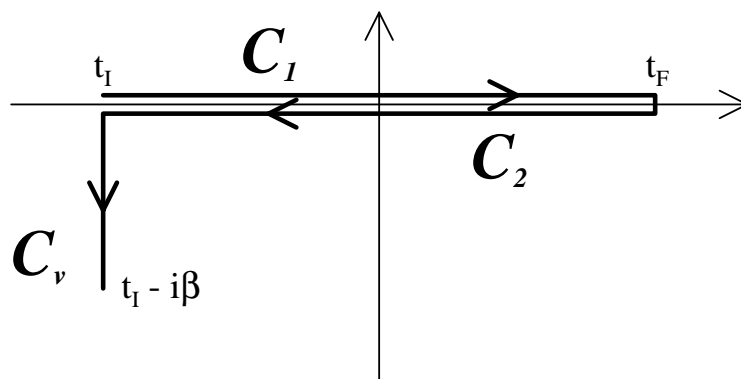


Figure 2

$$\begin{array}{c}
\text{---} \\
+ \quad \text{---} \times \text{---} \quad -i\mu^2 \\
+ \quad \text{---} \times \text{---} \times \text{---} \quad -i\mu^2 \quad -i\mu^2 \\
\vdots
\end{array}$$

Figure 3

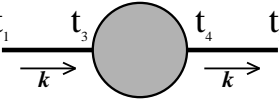
$$G_0(t_1, t_3; \omega_k) (-i\Sigma(t_3, t_4; \mathbf{k})) G_0(t_4, t_2; \omega_k)$$


Figure 4

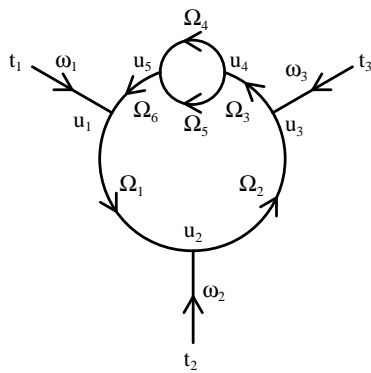


Figure 5